RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. MID SEMESTER EXAMINATION, SEPTEMBER 2012

THIRD YEAR

MATHEMATICS (Honours)

Paper : VI

Date : 13/09/2012

Time : 2 pm – 4 pm

Full Marks : 50

[Use separate answer-books for each group]

Group–A

Answer **any three** of the following: 3x5

1. a) Let $f:\mathbb{R}^n \to \mathbb{R}^m$ be continuous at $v_0 \in \mathbb{R}^n$. Show that f is locally bounded at v_0. 2

   b) Let $f(x,y)$ have continuous first order partial derivatives. Prove that the directional derivative $\frac{\partial f}{\partial \xi_\alpha}$ is a linear combination of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. 3

2. Let U be an open subset of $\mathbb{R}^n$ and let $f:U \to \mathbb{R}^m$ be a function defined by $f(v) = (f_1(v), f_2(v), \cdots, f_m(v))$ when each $f_i : U \to \mathbb{R}$, $1 \le i \le m$. Let f be differentiable at a point v of U. Prove that each $\frac{\partial f_i}{\partial x_j}(v)$ exists, $1 \le i \le m, 1 \le j \le n$. 5

3. Let $f:S \to \mathbb{R}$ where S is an open subset of $\mathbb{R}^2$ and let $(a,b) \in S$. Let (i) f_x and f_y exist in some neighbourhood of (a, b) and let (ii) $f_{xy}(x,y)$ is continuous at (a, b). Prove that $f_{xy}(a,b)$ exists and $f_{cy}(a,b) = f_{yx}(a,b)$. 5

4. Transform $Z_{xx} - 2Z_{xy} + Z_{yy} = 0$, taking $u = x + y$, $v = \frac{y}{x}$ for new independent variables and $w = \frac{z}{x}$ for the new function $w \equiv w(u,v)$. 5

5. If u is a homogeneous function of degree $n(\neq 1)$ in x, y, z and if $u = f(\xi, \eta, \rho)$ where $\xi = u_x, \eta = u_y, \rho = u_z$ and if all concerned second order partial derivatives are continuous, prove that $\xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \rho \frac{\partial u}{\partial \rho} = \frac{nu}{n-1}$. 5

6. Let $u = f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Let a new variable t be introduced by setting $x = t,\ y = t$. Show that at $t = 0$, $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$ does not hold. Explain why the chain rule fails here. 2+3

Group–B

7. Answer **any one** question: 1x5

a) POP' is a variable diameter of the ellipse $z=0,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and a circle is described in the plane $PP'ZZ'$ on PP' as diameter. Prove that as PP' varies, the circle generates the surface $\left(x^2+y^2+z^2\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=x^2+y^2$. 5

b) The enveloping cone of the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is cut by the plane $z=0$ in a parabola. Show that its vertex lies on the surface $z=\pm c$. 5

8. Answer **any one** question: 1x5

a) If $\vec{A}=(4xy-3x^2z^2)\vec{i}+2x^2\vec{j}-2x^3z\vec{k}$, prove that $\int_c \vec{A}\cdot d\vec{r}$ is independent of the curve joining two given points. Show that there is a differentiable function such that $\vec{A}=\vec{\nabla}\phi$ and find it.

b) Evaluate $\iint_S \vec{A}\cdot\vec{n}dS$, if $\vec{A}=y\vec{i}+2x\vec{j}-z\vec{k}$ and S is the surface of the plane $2x+y=6$ in the first octant cut off by the plane $z=4$.

Group–C

Answer **any one** from *Q. 9, 10 and Q.11*

9. a) Define momental ellipsoid at a point. Show that the momental ellipsoid at a point on the rim of a hemisphere is $2x^2+7(y^2+z^2)-\frac{15}{4}zx=\text{constant}$. 2+6

b) A rod of length $2a$ is suspended by a string, of length l attached to one end. If the string and the rod revolve about the vertical with uniform angular velocity and their inclinations to the vertical be θ and φ respectively, then show that $\frac{3l}{a}=\frac{(4\tan\theta-3\tan\varphi)\sin\varphi}{(\tan\varphi-\tan\theta)\sin\theta}$. 7

10. a) Define the terms centre of suspension , centre of oscillation of a compound pendulum.

A solid homogeneous cone of height h and semi-vertical angle α oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is $\frac{h}{5}(2+3\tan^2\alpha)$. 3+5

b) A wire is in the form of a semi-circle of radius a. Show that, at an end of its diameter, the principal axes in its plane are inclined to the diameter at angles $\frac{1}{2}\tan^{-1}\frac{4}{\pi}$ and $\frac{\pi}{2}+\frac{1}{2}\tan^{-1}\frac{4}{\pi}$. 7

11. Answer **any two** questions: 5+5

a) If the planet were suddenly stopped in its orbit, supposed circular, then show that it would fall into the sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.

b) Prove that for a parabolic orbit, the time taken to move from the vertex to a point distant r from the focus is $\frac{1}{3\sqrt{\mu}}(r+l)\sqrt{2r-l}$, where $2l$ is the latus rectum.

c) A body is describing an ellipse of eccentricity e under the action of a force tending to a focus. If the velocity of the body be doubled when it is at one end of the minor axis, then prove that the new path is a hyperbola of eccentricity $\sqrt{9-8e^2}$.
